OUTLINE OF THE PROPOSED TOPIC OF RESEARCH

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1. Proposed Topic of Research

Classes of Analytic Functions Involving Fractional Integral Operators and its Applications.

2. Objective of the Proposed Research

Objective of this study is to find gaps by review of literature and try to fill these gaps by introducing some new classes of analytic functions defined in the open unit disk and discuss their geometric properties.

(1) To define new class and find extreme points, redius of starlikeness, derive Fekete-Szegö type coefficient inequalities and its application with fractional operators by expressing it in convolution form.
(2) To study the inclusion property and other geometric properties by introducing some new classes of analytic functions defined using Komatu Integral.
(3) To discuss geometric properties by defining some new classes of analytic function using subordination and differential subordination.
(4) To obtain sufficient conditions of starlikeness, convexity and close-to-convexity, alpha-convexity, spiralikeness for the newly defined classes of analytic functions by using the principles of differential subordination.
(5) To find the applications of the above defined classes.

3. Area of Proposed Research

Geometric Function Theory and Fractional Calculus

Introduction:

Geometric Function Theory is that branch of complex analysis, which deals and studies the geometric properties of the analytic functions. That is Geometric function Theory is an area of Mathematics characterized by an intriguing marriage between geometry and analysis. Its origins date from the 19th century but new applications arise continually. Interest in Geometric Function Theory has experienced resurgence in recent decades as the methods of algebraic geometry and function theory on compact Riemann surfaces have found relevance in constructing 'finite-gap' solutions to non-linear integrable system, a growing, specialized area of Mathematics with many connections to Mathematical Physics.

The geometric theory of complex variable functions was set as a separate branch of complex analysis in the 20-th century when appeared the first important papers in this domain, owed to P. Koebe [17], I.W. Alexander [3], L. Bieberbach [7].

The univalent function notion occupies a central role in geometric theory of analytic functions, first paper dating since 1907 owed to P. Koebe. The study of univalent functions was continued by Plemelj, Gronwall and Faber. Even new developments in the constructive approach to linear and non-linear boundary value and initial value problems using spectral analysis [5] are likely to lead to a role for Geometric Function Theory in the solution of a wide range of partial differential equations.

Geometric Function Theory is a classical subject. Yet it continue to find new applications in an ever-growing variety of areas such as modern mathematical physics,

more traditional fields of physics such as fluid dynamics, non-linear integrable systems theory of partial differential equations. Geometric Function Theory is not as old as other branches of Mathematics. The first stirring of function theory is found in the 18th century with Euler. Modern function theory was developed in the 19th century.

Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The basic mathematical ideas of fractional calculus (integral and differential operations of non integer order) were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. The studies have also found its use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electerical networks, electromagnetic theory and probability.

4. Review of Literature

The geometric theory of complex variable functions was set as a separate branch of complex analysis in the XX-th century when appeared the first important papers in this domain, owed to P. Koebe [17], I.W. Alexander [3], L. Bieberbach [7]. The univalent function notion occupies a central role in geometric theory of analytic functions, first paper dating since 1907 owed to P. Koebe. The study of univalent functions was continued by Plemelj, Gronwall and Faber. Now exist many treated and monographs dedicated to univalent functions study. Among them we recall those of P. Montel, Z. Nehari, L.V. Ahlfors [1], Ch. Pommerenke [34], A.W. Goodman [11], P.L. Duren, D.J. Hallenbeck, T.H. MacGregor [13], S.S. Miller, P.T. Mocanu [23] and P.T.Mocanu, T. Bulboacoa, Gr. St. Salagean [24].

The theory of univalent functions is one of the most beautiful subjects in Geometric Function Theory. Its origin (apart from the Riemann mapping Theorem) can be traced to the 1907 paper of Koebe [17], to Gronwall's proof of the Area Theorem in 1914 and to Bieberbach's estimate for the second coefficient of normalized univalent functions in 1916 and its consequences. By then, univalent function theory was a subject in its own right. The Mathematical Romanian School brought her valuable contribution in the geometric theory of univalent functions. Among them we mention two personalities from Cluj, namely G.C Aalugareanu and P. T. Mocanu. G. C Alugareanu, the creator of the romanian school's theory of univalent functions, was the first mathematician who obtain in 1931 the necessary and sufficient conditions for univalence in the open unit disc. The researches initiated by G. C. Alugareanu are continued by P. T. Mocanu, who obtained important results in the geometric theory of univalent functions: introducing alpha-convexity, getting univalence criteria for nonanalytic functions, development in collaboration with S. S. Miller the method of differential subordinations, and most recently the theory of differential superordinations. The method of differential subordinations has an important role in a much easier demonstration of already known results, as well as in many other new obtained results.

4.1. **Classes of analytic functions.** The class S^* of starlike function was first studied by Alexander [3]. In 1936 Rebertson [36] introduce the class $S^*(\rho)$ of starlike function of order ρ. In 1936 ,Robertson [36] introduced the class $C(\rho)$ of convex function of order ρ , $0 \leq \rho < 1$. In 1969, P.Mocannu [25] introduced the concept of alpha- convex and alpha - starlike functions. In 1973, [22] it was proved that all α-convex function are convex if $\alpha \geq 1$ and starlike if $\alpha < 1$. The class of close - to - convex function introduced by Kaplan [15] in 1952 and denoted by K. The class K has been introduced independently and in a quite different manner by Biernacki [8]. The close-to-convex functions are univalent, (see [15]) and the class K proved the most useful subclass of S .We note that convex and starshaped domains are close-to-convex. The class K is of considerable importance in as much as it contains most of the known subclasses of S, this can be summarized as: $C \subset S^* \subset K \subset S$.

Noor [30] Considered a new subclass C^* of univalent function. The functions in C^* is called quasi-convex functions and $C \subset C^* \subset K \subset S$. It is shown (see [30]) $f \in C^*$ iff $zf' \in K$. The function f called α- quasi -convex function has been defined and its properties were studied in [29]. The class of α- quasi -convex functions is denoted by Q_α. In 1970 , Robertson [37], introduced the concept of Quasi Subordination. In 1934/35 Noshiro and Warchawaski obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $Re\{f'(z)\} > 0$ for all $z \in E$, then f is close-to-convex and hence univalent in E.

Al-Amiri and Reade [2], in 1975, have shown that for $\alpha \leq 0$ and also for $\alpha = 1$, the functions $f \in A$, satisfying the differential inequality $R[I(\alpha, f(z))] > 0; z \in E$, are univalent in E. In 2005, Singh, Singh and Gupta [46] proved that for $0 < \alpha < 1$, functions $f \in A$, satisfying the differential inequality $R[I(\alpha, f(z))] > \alpha; z \in E$, are univalent in E. The univalence of the above problem is still open for $\alpha > 1$.

Janowski introduced the class $P[A, B]$, for A and B. Classes $S^*[A, B]$ and $K[A, B]$ are introduced by K. S. Padmanabhan [32]. Ma and Minda [21] introduced the classes $S^*(\phi)$ and $C(\phi)$. The class $S^*(\phi)$ reduce to the class $S^*[A, B]$ of Janowski Starlike function. For $0 < \alpha \leq 1$, Parvatham introduced and studied the $S^*[\alpha]$. Following Ma and Minda [21], the authors defined a more general class of starlike functions of complex order $S_b^*(\phi)$ [35] and convex functions of complex order $C_b(\phi)$ [35].

$R(\alpha, \beta)$ is the class of functions α-Prestarlike of order β and was introduced by Sheil - Small, Silverman and Silva [44]. $C(\alpha, \beta)$ is called the class of functions of α -preconvex of order β .

Definition 4.1.1. If f (z) and g(z) are given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then *Convolution or Hadamard Product* of f(z) and g(z) is

$$h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

Definition 4.1.2. *Faltung Product*$(f * *g)$ defines as

$$J(z) = f(z) * *g(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n$$

4.2. **Fractional Operators.**

Definition 4.2.1. Let f(z) be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative* of f of order λ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \qquad (0 \leq \lambda < 1)$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $log(z-\zeta)$ is real for $(z-\zeta) > 0$

Definition 4.2.2. *Fractional Derivative of order* $(n+\lambda)$ [49] is defined as:

$$(D_z^{n+\lambda}) f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1; n \in N_0)$$

Definition 4.2.3. Let f(z) is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Then the *generalized fractional derivative of order* λ is defined for a function f(z) by (see, e.g. [48])
(4.2.1)

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \left\{ z^{\lambda-\mu} \displaystyle\int_0^z (z-\zeta)^{-\lambda} {}_2F_1 \left(\mu-\lambda, -\nu; 1-\lambda; 1-\frac{\zeta}{z} \right) f(\zeta) d\zeta \right\}, \\ (0 \leq \lambda < 1) \\ \dfrac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), (n \leq \lambda < n+1, n \in N) \end{cases}$$

and $f(z) = O(|z|^k), (z \to 0, k > max\{0, \mu-\nu-1\} - 1)$
It follows at once from the above definition that

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) := D_z^{\lambda} f(z), (0 \leq \lambda < 1)$$

Furthermore, in terms of gamma function, we have:

$$J_{0,z}^{\lambda,\mu,\nu} z^{\rho} := \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+1)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu},$$

$(0 \leq \lambda;\ \rho > max\{0.\mu-\nu-1\} - 1)$

Definition 4.2.4. *Fractional Integral of order* λ [49] is defined for a function f, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta (\lambda < 0)$$

Where f(z) is an analytic functions in a simply connected region of the complex z-plane containing the origin (z=0) , and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring log $(z-\zeta)$to be real when $z-\zeta > 0$.

Definition 4.2.5. The operator $I_{0,z}^{\alpha,\beta,\eta}$ is a *generalization of the fractional integral operator* $I_{0,x}^{\alpha,\beta,\eta}$ introduced by Saigo [39]. For real numbers $\alpha > 0, \beta$ and η, the fractional integral operator defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} {}_2F_1 \left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z} \right) f(\zeta) d\zeta$$

Where f(z) is an analytic function in a simply connected region of the z- plane containing the origin , with the order $f(z) = 0(|z|^{\epsilon})(z \to 0)$ where $\epsilon > max\{0, \beta - \eta\} - 1$ and the multiplicity of$(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to by real when $(z-\zeta) > 0$.

Definition 4.2.6. The *fractional integral operator* $J_{0,z}^{\alpha,\beta,\eta}$ [16] is introduced by Srivastava and defined as:

Let $\alpha > 0$, $\min(\alpha+\eta, -\beta+\eta, -\beta) > -2$ and $3 \geq \frac{\beta(\alpha+\eta)}{\alpha}$ then the fractional integral operator defined by

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z)$$

Definition 4.2.7. *Ruscheweyh Derivatives of* $(n+p-1)^{th}$ *order* [26] denoted by $D^{n+p-1} : A(p) \to A(p)$ and defined by

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad (f(z) \in A(P))$$

$$= \frac{z^p (z^{n-1} f(z))^{n+p-1}}{(n+p-1)!}$$

The symbol D^{n+p-1} when p =1 was introduced by Ruscheweyh [38], and the symbol D^{n+p-1} was introduced by Goel and Sohi [10].

recurrence relation [26]:

$$z(D^{n+p-1} f(z))' = (n+p) D^{n+p} f(z) - n D^{n+p-1} f(z)$$

Definition 4.2.8. *Generalized Ruscheweyh derivative* introduced by Goyal and Goyal [33] is defined as

$$J^{\lambda,\mu} f(z) := \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\mu+1)\Gamma(\nu+2)} z J_{0,z}^{\lambda,\mu,\nu} (z^{\mu-1} f(z))$$

$$= z + \sum_{n=2}^{\infty} a^n B^{\lambda,\mu}(n) z^n$$

where

$$B^{\lambda,\mu}(n) := \frac{\Gamma(n+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(n+\nu+1)}{\Gamma(n)\Gamma(n+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}$$

For $\mu = \lambda$, generalized Ruscheweyh derivative $J^{\lambda,\mu}$ reduces to the ordinary Ruscheweyh derivative D^{λ} of order λ.

Convolution:

$$J_{0,z}^{\lambda,\mu,\nu} z^p = {}_2F_1(\mu+1, \nu+2; \nu+2+\mu-\lambda; z) * f(z)$$

Definition 4.2.9. Let the function $f \in A$, for $m, n \in N_0 = N \cup \{0\}$, $0 \leq \lambda_1 \leq \lambda_2$, E. A. Eljamal and M. Darus [9] define the operator

$$D_{\lambda_1,\lambda_2}^{n,m} f(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m C(n,k) a_k z^k$$

Which is generalized Ruscheweyh derivatives operator.

Definition 4.2.10. For $f(z) \in A$, *p analogue of Salagean Operator* introduced by Shenen [45] which is the extension of Salagean Operator defined by Salagean [40]

$$D_{\lambda}^{\eta,p} f(z) = z^p + \sum_{k=n+p}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\lambda\right]^{\eta} a_k z^k, \quad z \in E$$

$$D_{\lambda}^{0,p} f(z) = f(z)$$

$$D_{\lambda}^{1,p} f(z) = (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) = D_{\lambda} f(z)$$

$$D_\lambda^{n,p} f(z) = D_\lambda(D_\lambda^{n-1,p} f(z))$$

For $\lambda = 1$, p=1 this p analogue reduce to be Salagean Operator introduced by Salagean [40]

Note that,

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = \{0\} \cup N$$

Definition 4.2.11. Owa and Srivastava [31] introduced the operator $\Omega^\lambda : A \to A$ defined by

$$\Omega^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \quad (\lambda \neq 2, 3, 4....)$$

The operator Ω^λ is called *Owa -Srivastava Operator* [49].

Definition 4.2.12. *AL- Oboudi Differential Operator* [4] defines as:

$D^0 f(z) = f(z)$

$D^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z). \qquad \lambda \geq 0$

$D^n f(z) = D_\lambda(D^{n-1} f(z))$

$\quad = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k$

If we put $\lambda = 1$, we have Salagean operator.

Definition 4.2.13. Let $n \in N \cup \{0\}$ and $\lambda \geq 0$. By Ω_λ^n we denote the *Najafzadeh operator* [28] $\Omega_\lambda^n : N \to N$ defined by

$$\Omega_\lambda^n f(z) = (1-\lambda) S^n f(z) + \lambda R^n f(z)$$

$S^n f \ \to$ Salagean Operator [40]

$R^n f \ \to$ Ruscheweyh differential operator [38]

For univalent function

$$\Omega_\lambda^n f(z) = z - \sum_{k=+1}^{\infty} \left[k^n (1-\lambda) + \lambda \binom{k+n-1}{n} \right] a_k z^k \tag{4.2.2}$$

Also $\Omega_\lambda^0 f(z) = f(z)$

Definition 4.2.14. *Schawarzian Derivative of w with regards to z* [42]

$$\sigma_3(w) = \{w, z\} = \left(\frac{w''}{w'}\right) - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

And

$$\sigma_{n+1}(w) = \sigma_n(w)' - (n-1)\frac{w''}{w'}\sigma_n(w)$$

ie

$$\sigma_4(w) = \frac{w''''}{w'} - 6\frac{w'''w''}{w'^2} + 6\left(\frac{w''}{w'}\right)^3$$

Definition 4.2.15. For function $f(z) \in A$, *Libra integral operator* defined as

$$L\{f(z)\} = \frac{2}{z}\int_0^z f(t)dt$$

This operator introduced by Libra [19].

Definition 4.2.16. For a function $f(z)$ belonging to the class A, *generalized Libra integral operator* defined as:

$$L_c\{f(z)\} = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \quad (C \geq 0)$$

The operator L_c, when $C \in N$ was introduced by Bernardi [6].

Recurrence relation:

$$(1+c)f(z) = cL_c f(z) + z(L_c f(z))'$$

Definition 4.2.17. *Bernardi Operator* (J_μ) [27] defined as,

$$(J_\mu)f(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \equiv H_1^2(\mu+1,1;\mu+2)f(z)$$

Note that the operator was studied earlier by Libra [20].

Definition 4.2.18. *Komatu Integral Operator* [41], [18] is defined as

$$K_{c,p}^{\delta} f(z) = \frac{(c+p)^\delta}{\Gamma(\delta)z^c}\int_0^z t^{c-1}(log\frac{z}{t})^{\delta-1}f(t)dt$$

For p-valent function

$$K_{c,p}^{\delta} f(z) = z^p + \sum_{k=1}^{\infty}\left(\frac{c+p}{c+P+k}\right)^\delta a_{p+k}z^{p+k}$$

Definition 4.2.19. *Generalization of Komatu integral operator* has been given by T. O. Salim [41] directly for p-valent functions

$$J_{c,p,\lambda}^{m,\delta} f(z) = z^p + \sum_{k=1}^{\infty}\left[\left(\frac{c+p}{c+p+k}\right)^\delta\left(1+\frac{\lambda k}{p}\right)\right]^m a_{p+k}z^{p+K}$$

For m=1,$\lambda = 0$ it reduces to Komatu Integral Operator.
$J_{c,p,\lambda}^{0,0} f(z) = f(z)$
$J_{c,p,\lambda}^{1,\delta} f(z) = (1-\lambda)K_{c,p}^{\delta} f(z) + \frac{\lambda z}{p}(K_{c,p}^{\delta} f(z)) = J_{c,p,\lambda}^{\delta} f(z)$
$J_{c,p,\lambda}^{m,\delta} f(z) = J_{c,p,\lambda}^{\delta}(J_{c,p,\lambda}^{m-1,\lambda} f(z))$
Remarks [41]:
$J_{c,1,0}^{1,1} \equiv L_c$ Barnardi-libra-Livingston operator [6]
$J_{c,1,\lambda}^{m,0} \equiv D_\lambda^m$ Generalized Salagean operator [4]
$J_{c,1,0}^{m,0} \equiv D^m$ Salagean operator [40]
Recurrence Relation [41] **:**

$$z\{J_{c,p,\lambda}^{m,s} f(z)\} = (c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) - c\,J_{c,p,\lambda}^{m,s} f(z)$$

4.3. **Some definitions and some standard classes of univalent functions.**

Definition 4.3.1. A complex function is said to be *analytic function* on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with *analytic function.* A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{4.3.1}$$

which are analytic in the open unit disk $E = \{z : z \in Cand|z| < 1\}$.

Definition 4.3.2. A function f analytic in a domain D is said to be *univalent function* there if it does not take the same value twice, that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D. In other words f is one-to-one (or injective) mapping of D onto another domain. S be the subclass of A consisting of all univalent functions in E.

Definition 4.3.3. The class P is the set of all functions of the form

$$P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... + c_n z^n ... = 1 + \sum_{n=1}^{\infty} c_n z^n$$

that is analytic in E and such that for $z \in E$,

$$Re\{p(z)\} > 0 \;\; ie \;\; p(z) \in P \text{ iff } p(z) \prec \frac{1+z}{1-z}$$

Any function in P is called a *functions with positive real part* in E.

It should be noted that $p(z)$ is not necessarily required to be univalent. Thus $p(z) = 1 + z^n$ is in P for any integer $n \geq 0$, but if $n \geq 2$ the function is not univalent.

Definition 4.3.4. The set E is said to be *Starlike* with respect to a point w_0 if the linear segment joining any point of E and w_0 lies entirely in E. The class S^* was first studied by Alexander [3] let $f \in A$, $f(0) = 0$ and f is univalent in E , Then f maps E onto a starlike domain w. r. to w_0 iff

$$Re\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in E$$

Definition 4.3.5. In 1936 Robertson [36] introduce the class $S^*(\alpha)$. If $f \in A$ is said to be in the class $S^*(\alpha)$ of *univalent starlike functions of order* α if

$$Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad (0 \leq \alpha < 1, \; z \in E)$$

We write $S^*(0) = S^*$,the class of univalent starlike in E.

Definition 4.3.6. The set E is said to be convex if it is starlike with respect to each of its points; that is, if the linear segment joining any two points of E lies entirely in E. A *convex function* is one which maps the unit disk conformally onto a convex domain. The class of all convex functions denoted by C let $f \in A$ and let f be univalent in E . Then f maps E onto a convex domain iff

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \quad z \in E$$

This condition was first stated and Study by Sheil-small [47].

Definition 4.3.7. In 1936, Robertson [36] introduced the class $C(\rho)$ of *convex function of order* ρ, $0 \leq \rho < 1$ which are defined by

$$C(\rho) = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \rho, \; 0 \leq \rho < 1, \; z \in E \right\}$$

In particular $C(0) = C$, Thus $\qquad C \subset S^* \subset S$

Definition 4.3.8. A function $f \in A$ is said to be *close- to- convex* iff

$$Re\left[\frac{zf'(z)}{g(z)}\right] > 0, z \in E$$

for some $g \in S^*$ or Equivalently if

$$Re\left[\frac{zf'(z)}{G'(z)}\right] > 0, \quad z \in E \text{ for some } G \in C$$

This class introduced by Kaplan in 1952 [15] and denoted by K.

The class K has been introduced independently and in a quite different manner by Biernacki [8]. The close-to-convex functions are univalent, see [15] and the class K proved the most useful subclass of S .We note that convex and starshaped domains are close-to-convex. The class K is of considerable importance in as much as it contains most of the known subclasses of S, this can be summarized as:

$$C \subset S* \subset K \subset S$$

Definition 4.3.9. In 1969, Mocannu [25] introduced the concept of *alpha- convex* and *alpha - starlike functions.* A function f is said to be α - convex in the open unit disk E if it is analytic,

$$\frac{f(z)f'(z)}{z} \neq 0 \quad \text{and} \quad Re[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}] > 0, \quad z \in E$$

The set of all such function is denoted by M_α

If $\alpha = 1$, then α-convex function is convex and if $\alpha = 0$ then α-convex function is starlike.

In 1973 [22], it was proved that all α-convex function are convex if $\alpha \geq 1$ and starlike if $\alpha < 1$.

Definition 4.3.10. Noor [30] considered a new subclass C^* of univalent function ie $f \in A$ belongs to C^* iff f is analytic in E and is such that there exists $z \in C$ satisfying

$$Re\left[\frac{(zf'(z))'}{g'(z)}\right] > 0, \quad z \in E$$

The function in C^* is called *quasi-convex* and

$$C \subset C^* \subset K \subset S$$

Note: $f \in C^*$ iff $zf'(z) \in K$.(see [30])

Definition 4.3.11. The function f called α-*quasi convex function* has been defined and its properties studied in [29]. A function f, analytic in E, is said to be α- quasi convex iff there exists $g \in C$ such that, for α real and positive

$$Re\left[(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] > 0, \quad z \in E$$

The Class of α- quasi convex function is denoted by Q_α

Definition 4.3.12. For functions f and g, analytic in Δ, we say that f is *subordinate* to g if there exists a Schwarz function $w(z)$ analytic in Δ, with $w(0) = 0$ and $|w(z)| < |z| < 1$ $(z \in \Delta)$ Such that

$$f(z) = g(w(z)) \quad (z \in \Delta)$$

We denote this *subordination* as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \Delta)$ (see [12])

Definition 4.3.13. A sequence$\{b_k\}_{k=1}^{\alpha}$ of complex number is said to be a *subordinating factor sequence*(see [12]) if for the function$f(z) \in A$ is analytic, univalent and convex in U. We have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \quad (z \in E;\ a_1 = 1)$$

Definition 4.3.14. In 1970 , Robertson [37] introduced the concept of *Quasi Subordination.* For two analytic functions f and g , the function f is Quasi Subordination to g written as

$$f(z) \prec_q g(z)$$

If there exists analytic functions ϕ and w, with $|\phi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = \phi(z)g(w(z))$$

observe that $\phi(z) = 1$ then $f(z) = g(w(z))$ so that $f(z) \prec g(z)$

Note:

Also notice that if $w(z) = (z)$ then $f(z) = \phi(z)g(z)$ and it is said that f is Majorised by g and written as

$$f(z) << g(z) \quad \text{in} E$$

4.4. Major subclasses in literature defined using subordination.

Definition 4.4.1. Janowski Class $P[A, B]$(see [14],):

Janowski introduced the class $P[A, B]$, for A and B ,$-1 \leq B < A \leq 1$, a function p analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$ if

$$p(z) \text{ is subordinate to } \frac{1 + Az}{1 + Bz} \quad ie \quad p(z) \prec \frac{1 + Az}{1 + Bz}$$

In particular $P[A, B] \subset P[1, -1] = P$

We also note that

$$p \in P[A, B] \text{ iff } |p(z) - \frac{1 - AB}{1 - B^2}| < \frac{A - B}{1 - B^2}, (B \neq \pm 1, \ z \in E)$$

In addition it is known that $P[A, B]$ is a convex set.Furthermore, it can easily be shown that $p \in P[A, B]$ iff there exists a function $h \in P$ such that

$$p(z) = \frac{(1 - A) + (1 + A)h(z)}{(1 - B) + (1 + B)h(z)}$$

Special selection of A and B lead to a familiar sets defined by inequalities. Some of these are given below under condition $p(0) = 1$ and $0 \leq \rho < 1$

(i) $P[1, -1]$ is the class P of function p with positive real part

(ii)$P[1 - 2\rho, -1]$ is the class $P(\rho)$ of function p with $Re\{p(z)\} > \rho$ in E

(iii) $P[1, 0]$ is the class of function defined by $|p(z) - 1| < 1$

Definition 4.4.2. Classes Introduced By Ma And Minda:

Ma and Minda [21] introduced the classes $S^*(\phi)$ and $C(\phi)$ by

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

When

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \ (-1 \leq B \leq A \leq 1)$$

The class $S^*(\phi)$ reduce to the class $S^*[A, B]$ of Janowski Starlike function.

Thus $S^*[A, B] = S^*\left(\frac{1+Az}{1+Bz}\right)$

Also $S^*[1, -1] = S^*\left(\frac{1+z}{1-z}\right) = S^*$

For $0 \leq \alpha < 1$, the class $S^*[1 - 2\alpha, -1]$ is the class $S^*(\alpha)$ of starlike function of order α .

An equivalent analytic description of $S^*(\alpha)$ is given by

$$S^*(\alpha) = \left\{ f \in A; Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ 0 \leq \alpha < 1 \right\}$$

for $0 \leq \alpha < 1$

$$S^*(\alpha) = S^*[1-\alpha, 0] = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \right\}$$

For $0 < \alpha \leq 1$ Parvathom introduced and studied the $S^*[\alpha]$
Where

$$S^*[\alpha] = S^*[\alpha, -\alpha] = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1+\alpha(z)}{1-\alpha(z)} \right\}$$

$$= \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \left| \frac{zf'(z)}{f(z)} + 1 \right|, z \in E, 0 < \alpha \leq 1 \right\}$$

(For Convex)
Ma and Minda introduced the class$C(\phi)$ by

$$C(\phi) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}$$

When

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1)$$

Class $C(\phi)$ reduce to the class $C[A, B]$ of Janowski Convex function ,
thus $C[A, B] = C\left(\frac{1+Az}{1+Bz}\right)$
Also $C = C[1, -1] = C\left(\frac{1+z}{1-z}\right)$
For $0 \leq \alpha < 1$,$C(\alpha) = C[1-2\alpha, -1]$ is this class of convex functions of order α

$$C(\alpha) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}$$

$$= \left\{ f \in A : Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha, \ 0 \leq \alpha < 1 \right\}$$

The transform $\int_0^z \frac{f(t)}{t} dt$ is called the Alexander transform of $f(z)$.
It is clear that $f \in C(\alpha)$ iff $zf' \in S^*_\alpha$ or equivalently $f \in S^*_\alpha$ iff Alexander transform of $f(z)$ is in $C(\alpha)$ (see [43])

5. Research Gap

It has been observed that the authors have been defining new classes and subclasses of analytic functions but there is a need to continue on developing the new classes and to investigate into the properties, however, continuous literature review is needed to support the work as the proposed work is dynamic in nature. This observation is influenced by its applications in the field of science and engineering which require supported mathematical tools for its increasing requirements. A vast scope is also found to highlight the applications of this work.

6. Methodology

The following methods are proposed to be used to undertake the proposed work:

1. Differential subordinations
2. Method of differential inequalities
3. Methods arising from the convolution theory.

These methods are also proposed to be used to find new theorems and to discuss the geometric properties of the defined classes.

Mathematical softwares like Mathematica and Matlab are proposed to be used to visualize these geometric properties.

7. Work Schedule

We are expecting to finish the work by 30 months from the date of commencement. Following is the tentative plan of action:

Phase 1: Introduction and review of literature

We are expecting to finish introduction and review of literature in (0-12)months. Continuous literature survey will be carried out to collect the information required during various stages of the proposed research till thesis writing.

Phase 2: Study of geometric properties of classes of analytic function

In this phase, we will find the gaps by review of literature in phase 1 and fill these gaps by introducing some classes of analytic function and study their geometric properties and finding some new results.

We are expecting to finish this phase in (6-24)months.

Phase 3: Conclusion and thesis writing

In this phase, all the work carried out in different phases will be documented in the form of thesis.

We are expecting to finish this phase in (24-30)months.

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